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THE STABILITY OF RIEMANN ELLIPSOIDS IN THE LYAPUNOV FORMULATION†

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The problem of the stability of the Riemann ellipsoids of a rotating uniform self-gravitating ideal liquid is considered within the framework of the Lyapunov definition of the stability of the form of equilibrium [1]. The regions such that almost all the ellipsoids belonging to it are unstable forms of equilibrium, specified in explicit analytical form, are determined in parameter spaces of the first and second families of Riemann ellipsoids. The proof is based on the general fact (which is formulated and justified separately) that, when an unstable equilibrium position of an autonomous system is stable with respect to a certain function, the trajectory of this system, which belongs to a certain manifold, is obtained, and also on a consequence of this fact, which has a constructive form. The stability of the form of ellipsoidal figures of equilibrium, with the exception of special cases of Maclaurin and Jacobi ellipsoids, the stability of the form of which was investigated by Lyapunov himself, has not been investigated previously in the literature.

1. INTRODUCTION. FORMULATION OF THE PROBLEM.

The problem of the existence and stability of equilibrium ellipsoids occupies a central place in the theory of ellipsoidal figures of equilibrium of a rotating liquid [2–4].

The problem of the existence of equilibrium ellipsoids was solved by Dirichlet and Riemann [3, 4]. Dirichlet showed [2, 3] that in a class of initial conditions, which satisfy the requirements that the surface of the liquid should be ellipsoidal and that its velocity field should be uniformly vortical ("Dirichlet's assumptions"), the initial infinite-dimensional system describing the dynamics of a uniform ideal incompressible self-gravitating rotating liquid, changes in a system of ordinary differential equations for the components of the vorticity ($2\omega_1(t)$, $2\omega_2(t)$, $2\omega_3(t)$), the semi-axes of the ellipsoid (a, b, c) and the components of the angular velocity (p, q, r) in a moving frame of reference.

We will present this system of ordinary differential equations in a form which will be most convenient later (obtained from the initial system [2] taking into account the condition for the liquid volume to be constant, which is a consequence of the equation of incompressibility: abc = const, where we can assume that const = 1 without loss of generality).

We have

$$\frac{d}{dt}(A_1p+A_2\omega_1)+q(C_1r+C_2\omega_3)-r(B_1q+B_2\omega_2)=0 \quad (pqr,\ \omega_1\omega_2\omega_3,\ ABC) \tag{1.1}$$

$$\frac{d}{dt}\left(\frac{\omega_1}{a}\right)-\frac{2a}{a^2+b^2}r\omega_2+\frac{2a}{a^2+c^2}q\omega_3+ \tag{1.2}$$

$$+\frac{2a(c^2-b^2)}{(a^2+c^2)(a^2+b^2)}\omega_2\omega_3=0 \quad (123,\ abc,\ pqr)$$

$$\ddot{a}\left(a+\frac{1}{b^2a^3}\right)+\frac{\ddot{b}}{a^2b^3}-\frac{2\dot{a}\dot{b}}{a^3b^3}-\frac{2\dot{a}^2}{a^4b^2}-\frac{2\dot{b}^2}{b^4a^2}=(R-\tilde{\omega}_z)\frac{1}{a^2b^2}-(P-\tilde{\omega}_x)a^2 \tag{1.3}$$

$$\ddot{b}\left(b+\frac{1}{a^2b^3}\right)+\frac{\ddot{a}}{b^2a^3}-\frac{2\dot{a}\dot{b}}{a^3b^3}-\frac{2\dot{a}^2}{a^4b^2}-\frac{2\dot{b}^2}{b^4a^2}=(R-\tilde{\omega}_z)\frac{1}{a^2b^2}-(Q-\tilde{\omega}_y)b^2 \tag{1.4}$$

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where

$$\tilde{\omega}_x = \frac{\ddot{a}}{a} - \omega_x = \frac{a^2 - c^2}{(a^2 + c^2)^2}(a^2 + 3c^2)q^2 + \frac{a^2 - b^2}{(a^2 + b^2)^2}(a^2 + 3b^2)r^2 -$$

$$-\frac{4(a^2 - c^2)c^2}{(a^2 + c^2)^2} q\omega_2 - \frac{4(a^2 - b^2)b^2}{(a^2 + b^2)^2} r\omega_3 + \frac{4a^2b^2}{(a^2 + b^2)^2}\omega_3^2 + \frac{4a^2c^2}{(a^2 + c^2)^2}\omega_2^2 \tag{1.5}$$

(xyz, abc, pqr)

$$A_1 = \frac{M}{5}\frac{(b^2 - c^2)^2}{b^2 + c^2}, \quad A_2 = \frac{4M}{5}\frac{b^2c^2}{b^2 + c^2} \quad (ABC,\ abc)$$

$$P = \frac{2}{a}\left[\frac{\partial}{\partial a}\hat{H}(a, b, c)\right] \quad (PQR,\ abc) \tag{1.6}$$

$$\hat{H} = \frac{3M}{4}\int_0^\infty \frac{d\lambda}{\sqrt{\varphi(\lambda)}}, \quad \varphi(\lambda) = (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)$$

and M is the mass of the liquid.

Here and everywhere henceforth we will assume $c = 1/(ab)$ in all the formulae.

If we take into account Dirichlet's conditions and the known form of the solution of Laplace's equation, Eqs (1.2) of system (1.1)–(1.4) follow from Helmholtz' equations for a vortex in moving axes, Eqs (1.1) follow from the theorem of angular momentum, and Eqs (1.3) and (1.4) follow from Euler's hydrodynamic equations in moving axes [2].

System (1.1)–(1.4) has three integrals: energy, angular momentum and constancy of the vorticity [2]

$$\frac{1}{2}(A_1p^2 + B_1q^2 + C_1r^2 + A_2\omega_1^2 + B_2\omega_2^2 + C_2\omega_3^2) + W + \frac{M}{10}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2) = \text{const} \tag{1.7}$$

$$(A_1p + A_2\omega_1)^2 + (B_1q + B_2\omega_2)^2 + (C_1p + C_2\omega_3)^2 = \text{const} \tag{1.8}$$

$$(\omega_1 / a)^2 + (\omega_2 / b)^2 + (\omega_3 / c)^2 = \text{const} \tag{1.9}$$

where

$$W = -\frac{2 \cdot 3}{5 \cdot 4} M^2 \int \frac{d\lambda}{\sqrt{\varphi(\lambda)}} \tag{1.10}$$

Riemann, on the basis of Dirichlet's investigations, determined [3, 4] the set of all equilibrium positions of system (1.1)–(1.4), to each of which there corresponds a certain steady motion of the mechanical system considered – the Riemann equilibrium ellipsoid.

Riemann showed [3] that this set is a combination of two "subsets": in ellipsoids of the first subset the angular velocity vector and the vorticity vector of the internal motions lie in one of the principal planes of the ellipsoid, and for the ellipsoids of the second subset these vectors are collinear and are directed along one of the axes [3].

Considering now *the problem of the stability of the equilibrium ellipsoids*, it is first necessary to note the fact that the formulation of this problem itself, like any other problem on systems with an infinite number of degrees of freedom, is non-unique. Two approaches to the consideration of the stability of the equilibrium ellipsoids are known in the classical literature, in addition to Poincaré's approach, that are close to the hydrodynamic method, which is not considered here.

In the first of these – Riemann's approach – the perturbation is assumed to satisfy Dirichlet's conditions, and the stability is determined in a natural way as the usual stability of the corresponding equilibrium position of system (1.1)–(1.4) [2, 3] (i.e. Lyapunov stability for systems with a finite number of degrees of freedom).

The second approach – Lyapunov's approach – assumes that the nature of the perturbation is, generally speaking, arbitrary. Stability here is understood as the stability of the form of equilibrium in accordance with the definition given by Lyapunov in his Master's Dissertation [1]: If for each sufficiently small number $\varepsilon > 0$ we obtain a number $\delta > 0$ such that for any perturbation, for which at the initial instant of time

$$|v(r)-v_0(r)|_{t=0}<\delta,\quad |\Omega-\Omega_0|<\delta,\quad l|_{t=0}<\delta \text{ and } \nabla|_{t=0}>\nabla_{\min}(l|_{t=0}) \tag{1.11}$$

the condition $l(t) < \varepsilon$ will hold, at least so long as the relation $\nabla(t) > \nabla_{\min}(l(t))$ is satisfied, then the steady motion corresponding to this equilibrium figure is stable.

Here $v(r)$ is the velocity field of internal motions, and $\Omega = (p, q, r)$ is the angular velocity of the ellipsoid in moving axes (the subscript 0 denotes the corresponding parameters for the equilibrium figure); the quantity l, introduced by Lyapunov and which he called [1] the distance of the perturbed surface of the liquid S from the unperturbed surface S_0, is defined as

$$l=\max_{P_0\in S_0}\left(\min_{P\in S}\rho(P,P_0)\right)$$

∇ is a quantity, which Lyapunov called the deviation, that corresponds to the volume of the part of the liquid in the perturbed configuration, bounded by the surfaces S and S_0 and situated above the surface S_0, and $\nabla_{\min}$ is a function of the distance l, which defines the minimum possible value of ∇ for the given value of l, which is possible for an actual liquid.

The method Lyapunov used to solve the problem was based on proofs that the functional of the changed potential energy in the steady rotation considered reaches a strict minimum [1]. Hence, he investigated the stability of the Maclaurin and Jacobi ellipsoids (Lyapunov apparently did not consider other equilibrium ellipsoids).

Lyapunov's method involves analysing that the sufficient condition for the stability of the form he formulated is satisfied. Unfortunately, from the technical (computational) point of view, the use of this method involves considerable difficulties. Moreover, when investigating Jacobi ellipsoids, Lyapunov, as is well known [1], only succeeded in proving the stability of the form with respect to a very "small" part of them (namely, those whose eccentricities lay in certain narrow limits). And, bearing in mind the fact that the Jacobi ellipsoids are the "simplest" special cases of Riemann ellipsoids, this implies that it will be extremely difficult, if generally possible, to obtain any results on the stability in respect of the general case of Riemann ellipsoids in this way. In this connection the question arises of whether it is possible to approach this solution of the stability problem from another angle, namely, by analysing whether (some) sufficient instability condition is satisfied rather than the sufficient stability condition.

In this paper we show that this approach does in fact enable one, generally speaking, to reach conclusions on the nature of the stability, in the sense of the Lyapunov definition (1.11), (to prove instability) of a considerable part (if not the major part, in the sense of the usual measure) of the Riemann ellipsoids.

It can be shown (see Section 2 below), that the conditional stability of form (in the class of perturbations which satisfy Dirichlet's conditions) in the sense of Lyapunov's definition (1.11) is equivalent to the partial stability of the equilibrium position of system (1.1)–(1.4), corresponding to the Riemann ellipsoid considered, with respect to part of the phase variables – the semi-axes (a, b) of system (1.1)–(1.4).

Hence, the definition of the set of equilibrium positions of system (1.1)–(1.4), which are unstable with respect to (a, b), automatically defines a set of Riemann ellipsoids, which are unstable forms of equilibrium in the class of perturbations that satisfy Dirichlet's conditions and, of course, the set of Riemann ellipsoids that are certainly unstable in the sense of Lyapunov's definition (1.11).

2. THE EQUIVALENCE OF THE CONDITIONAL STABILITY OF FORM AND OF THE STABILITY WITH RESPECT TO THE VARIABLES (a, b)

Consider the conditional stability of form – that property into which the property of stability of form of the Riemann ellipsoid transfers in the sense of the Lyapunov definition (1.11), given in [1], provided that the perturbations satisfy Dirichlet's assumptions.

In this case the whole time of motion of the liquid surface will be represented by the surface of an ellipsoid with semi-axes $a(t)$, $b(t)$ and $c(t)$.

Starting from this we will determine here the value of the distance l, distance by Lyapunov [1].

In the space $\mathbf{R}^3$ we will change to dimensionless elliptic coordinates (r, φ, θ)

$$x = ra_0\sin\varphi\sin\theta,\quad y = rb_0\cos\varphi\sin\theta,\quad z = rc_0\cos\theta$$

Then $r = 1$ is the equation of the unperturbed surface of the equilibrium ellipsoid, and φ and θ are coordinates on this surface.

The equation of the perturbed surface in these coordinates is

$$r(t,\varphi,\theta)=\left(\frac{a_0^2}{a^2(t)}\sin^2\varphi\sin^2\theta+\frac{b_0^2}{b^2(t)}\cos^2\varphi\sin^2\theta+\frac{c_0^2}{c^2(t)}\cos^2\theta\right)^{-1/2}$$

In the coordinates considered we will define the value $\tilde{l}$ of the "dimensionless" distance, corresponding to the value l. From Lyapunov's definition we have

$$\tilde{l}\leqslant L(a(t),b(t),c(t))=\max_{\varphi,\theta}|r(t,\varphi,\theta)-1| \tag{2.1}$$

since the distance from any point of the surface of the unperturbed ellipse with coordinates (φ, θ) to the point of the surface of the perturbed ellipsoid closest to it is certainly no greater than $|r(\varphi, \theta) - 1|$.

In is clear that the value of the quantity $L(a(t), b(t), c(t))$ corresponds to the value of the function $r(t, \varphi, \theta)$ for those values of φ and θ for which the function

$$\frac{1}{r^2(t,\varphi,\theta)}=\frac{a_0^2}{a^2(t)}\sin^2\varphi\sin^2\theta+\frac{b_0^2}{b^2(t)}\cos^2\varphi\sin^2\theta+\frac{c_0^2}{c^2(t)}\cos^2\theta \tag{2.2}$$

takes an extremum value (with respect to φ and θ). Differentiating function (2.2) with respect to φ and θ, we conclude that both derivatives will be zero only in each of six possible cases:

1) $\varphi = 0$, $\theta = \pi/2$, 2) $\varphi = 0$, $b_0^2/b^2(t)=c_0^2/c^2(t)$, 3) $\varphi = \pi/2$, $\theta = \pi/2$, 4) $\varphi = 0$, $a_0^2/a^2(t)=c_0^2/c^2(t)$, 5) $\theta = 0$, 6) $\theta = \pi/2$, $a_0^2/a^2(t)=b_0^2/b^2(t)$.

From the values function (2.2), in each of these cases we obtain

$$\tilde{l}(t)\leqslant\mu,\quad \mu=\max\{(a(t)-a_0)/a_0,\ (b(t)-b_0)/b_0,\ (c(t)-c_0)/c_0\} \tag{2.3}$$

On the other hand, for points of the unperturbed surface lying on the semi-axes a_0, b_0 and c_0 (cases 3, 1 and 5), the distance to the points of the perturbed surface closest to it is equal to $(a(t) - a_0)/a_0$, $(b(t) - b_0)/b_0$ and $(c(t) - c_0)/c_0$, respectively.

$$\tilde{l}(t) \geqslant \mu$$

Hence, taking inequality (2.3) into account, we have

$$\tilde{l}(t) = \mu$$

It is obvious, further, that for any specified point M' of the perturbed surface the definition of the point $M_1 = M_1(M')$ of the unperturbed surface is such that

$$\rho(M_1,M')=\min_{M}\rho(M,M')$$

is independent of the choice of the system of coordinates. In exactly the same way, the definition of the point M_2 of the perturbed surface, for which

$$\rho(M_1(M_2),M_2)=\max_{M'}\rho(M_1(M'),M')$$

is also independent of it. From this, and taking into account the definition of the distance l, we have that the two points M_2 on the perturbed surface and $M_1(M_2)$ on the unperturbed surface, the distance between which defines the value of l, are independent of the choice of the system of coordinates (although this value itself will, generally speaking, change on changing from one system of coordinates to another, if distance is not preserved during this transition). Taking the above discussion into account, we obtain that in the initial Cartesian coordinates (x, y, z)

$$l(t) = \max\{|a(t) - a_0|,\ |b(t) - b_0|,\ |c(t) - c_0|\} \tag{2.4}$$

(This obvious fact also be proved differently.)

Further, it can be seen from the expressions for the derivatives of the function $r(t, \varphi, \theta)$ with respect to φ and θ, that

$$|dr(t;\varphi,\theta)/d\varphi| < M \quad |dr(t;\varphi,\theta)/d\theta| < M, \quad \forall t,\varphi,\theta$$

where

$$M = 3\max(a_0^2, b_0^2, c_0^2)/[\min(a,b,c)]^5 < 3\max(a_0^2, b_0^2, c_0^2)/[(1-\tilde{l})\min(a_0, b_0, c_0)]^5 =$$
$$= M^*(a_0, b_0, c_0, \tilde{l})$$

Hence the derivatives $dr/d\varphi$, $dr/d\theta$ for any t, φ and θ are bounded in modulus by the constant $M^*(a_0, b_0, c_0, \tilde{l})$.

It immediately follows from this that part of the perturbed liquid, which is below the unperturbed surface, comprises a cone $K(\tilde{l}, a_0, b_0, c_0)$ of height $\tilde{l}$ and the base of which is a circle of radius

$$\rho = \rho(\tilde{l}) = \sin(\tilde{l}/M^*)\tilde{l}/(\tilde{l} + 1 - \cos(\tilde{l}/M^*))$$

We then obtain that a function $\nabla^*_{\min}(\tilde{l})$ exists, equal to the volume of the cone $K(\tilde{l}, a_0, b_0, c_0)$, and such that

$$\nabla(t) > \nabla^*_{\min}(l(t)), \quad \forall t \leqslant 0 \tag{2.5}$$

The condition corresponding to condition (2.5) will obviously also be satisfied in the initial coordinates (x, y, z).

Further, taking expression (2.4) into account, we obtain, from the expressions for the components of the velocity field of the internal motions of the liquid for the case when the initial conditions satisfy Dirichlet's assumptions [2], that for each number $\tilde{\delta} > 0$ a number $\delta > 0$ is obtained such that from the conditions

$$|\Delta v_{\rm rel}| < \delta,\ |\Omega - \Omega_0| < \delta,\ l < \delta$$

it follows that

$$|\Delta\omega_i| < \tilde{\delta},\ |\Delta p| < \tilde{\delta},\ |\Delta q| < \tilde{\delta}\,|\Delta r| < \tilde{\delta},\ |\Delta a| < \tilde{\delta},\ |\Delta b| = \tilde{\delta},\ |\Delta c| < \tilde{\delta} \tag{2.6}$$

where ω_i, p, q, r, a, b are the phase variables of system (1.1)–(1.4), which describe the dynamics of the liquid Dirichlet ellipsoid, $\Delta\omega_i = \omega_i - \omega_{i0}$, etc. On the other hand, since the velocity field in the case when the initial conditions satisfy Dirichlet's assumptions depends continuously on $x = (\omega_i, p, q, r, a, b, \dot{a}, \dot{b})$, we have that, for each number $\delta > 0$ a number $\delta_1(\delta) > 0$ is obtained such that the condition $|x - x_0| > \delta_1$ implies the condition $|\Delta v_{\rm rel}| < \delta$.

Hence, and also taking into account expression (2.4) and the fact that, from the condition $|x - x_0| < \delta_1$ it automatically follows that

$$|\Delta p| < \delta_1,\ |\Delta q| < \delta_1,\ |\Delta r| < \delta_1;\ |\Delta a| < \delta_1,\ |\Delta b| < \delta_1,\ |\Delta c| < \delta_1$$

we obtain that for each number $\delta > 0$ a number $\delta_2 = \min(\delta; \delta_1(\delta)) > 0$ is obtained ($\delta_1(\delta)$ is defined above), such that the condition $|x - x_0| < \delta_2$ yields the following inequality.

$$|\Delta v_{\rm rel}| < \delta;\ |\Delta\Omega| < \delta,\ l < \delta \tag{2.7}$$

As a result we conclude that when the initial conditions satisfy Dirichlet's assumptions:

(1) the condition $\nabla > \nabla_{\min}(l(t))$ can be omitted in definition (1.11), since it is automatically satisfied by virtue of inequality (2.5);

(2) the conditions

$$|\Delta v_{\rm rel}(t = 0)| < \delta,\ |\Delta\Omega(t = 0)| < \delta,\ l(t = 0) < \delta$$

in definition (1.11) are equivalent to the condition $|x(t = 0) - x_0| < \delta$, by virtue of conditions (2.7) and (2.8);

(3) the conditions $l(t) < \varepsilon$ in definition (1.11) is equivalent to the condition $\Delta a < \varepsilon$, $\Delta b < \varepsilon$, $\Delta c < \varepsilon$ by virtue of expression (2.4).

Hence, recalling that $c = 1/(ab)$, property (1.11) in this case is equivalent to the stability of the ellipsoid z_0 as an equilibrium position f system (1.1)–(1.4) with respect to the functions a and b in Lyapunov's sense, which it was required to prove.

Hence, the question naturally arises here regarding the stability of the Riemann ellipsoids as equilibrium positions of system (1.1)–(1.4) with respect to part (a, b) of the variables of system (1.1)–(1.4).

Before considering this specific question, it is worth dwelling on some general facts.

3. REMARKS ON THE PROBLEM OF THE STABILITY WITH RESPECT TO A PART OF THE VARIABLES

Lyapunov [1] first formulated the problem of stability with respect to a part of the variables.

However, this problem was not investigated until much later, after more than half a century, by Rumyantsev and his successors (see the bibliography in [5]).

In these papers the basic principles of the method of solving the problem of stability with respect to a part of the variables were developed, founded on Lyapunov's ideas. The analysis carried out in these investigations of both Lyapunov's theorem itself and the majority of the other theorems of the method of Lyapunov functions showed that assertions close to them also held and were applicable to the problem of stability with respect to a part of the variables.

General theorems, which gave the sufficient conditions for stability, instability, asymptotic stability, uniform asymptotic stability, and asymptotic stability as a whole, with respect to a part of the variables, were formulated in [5], modifications of these theorems were considered, and the problem of their invertibility were investigated. Rumyantsev was the first to introduce into the literature the idea of the positive definiteness of a function with respect to a part of the variables and to determine the property of the infinitesimal upper limit with respect to a part of the variables.

The majority of the theorems of Rumyantsev's method of the problem of stability with respect to a part of the variables are of universal form and are applicable in the majority of cases of non-autonomous dynamical systems.

Their use does not require a knowledge of any a priori information of the properties of the system considered in each specific case. But in certain cases the presence of such information can facilitate a solution of the problem, constructed using theorems of general form.

Below we will show that it is precisely this situation that occurs when solving the problem of the stability of the equilibrium positions of system (1.1)–(1.4) with respect to the variables (a, b), to which the initial problem can be reduced (see Sections 1 and 2), that is the object of the present paper. In this case, on the basis of an analysis of Rumyantsev's theorem on stability with respect to a part of the variables, we can formulate an addition to this theorem, which represents a certain development of it for a particular class of system to which system (1.1)–(1.4) belongs. This also enables us (see below) to reach certain conclusions regarding the stability of the equilibrium positions of the system considered with respect to the variables (a, b).

System (1.1)–(1.4) is autonomous. Moreover, in the phase space of this system it turns out to be possible to determine in analytical from [7] regions such that almost all the equilibrium positions of system (1.1)–(1.4) pertaining to them are Lyapunov unstable. Following the approach employed here to solving the initial problem (see Section 1) we will further consider the stability of these equilibrium positions of system (1.1)–(1.4) with respect to (a, b).

We will first consider the general case of the autonomous system

$$\dot{x} = v(x), x \in \mathbf{R}^n;\ v_i(x_0) = 0,\ i = 1, \dots, n \tag{3.1}$$

with Lyapunov unstable equilibrium positions $x = x_0 = 0$. Suppose it is required to the determine the nature of the stability x_0 with respect to the variables $y = (x_1, \dots, x_k)$, $k < n$, $(x = (x_1 \dots x_n) = (y, z)$; $z = (z_1 \dots z_{n-k}) = (x_{k+1} \dots x_n))$.

Rumyantsev's theorem on stability with respect to a part of the variables [5, 6], which henceforth, for brevity, we will refer to as the "theorem of y-stability", allows of inversion for a class of dynamical systems of fairly general form [5]. If some system of the form (3.1) belongs to this class and if the y-positive-definite function $V(t, x)$, which satisfies the conditions of the theorem on y-stability for this system (3.1), allows of an infinitesimal upper limit with respect to y, then [5] the set $\{x: y = 0\}$ is stable, and necessarily invariant for system (3.1) considered.

Hence, the question naturally arises here as to what extent, for systems of the form (3.1), it is possible to satisfy simultaneously the following two properties: y-stability and invariance of the set $\{x: y = 0\}$. A consideration shows that when y-stability occurs, the set M, even if it is not invariant, i.e. completely filled with trajectories of system (3.1), at all events contains the whole the separate trajectories having points in as small a neighbourhood of x_0 as desired. The corresponding assumption, which can be formulated in this connection, can be extended to several more general cases. Suppose it is required to determine the stability of x_0 with respect to the set of functions $f_1(x), \dots, f_k(x)$, which are arbitrary in the sense that the existence of the diffeomorphism $x \to (f_1(x), \dots, f_k(x), z_1(x) \dots z_{n-k}(x))$ is not assumed here.

Proposition 1. Suppose x_0 is an unstable equilibrium position of the system $\dot{x} = v(x), x \in \mathbf{R}^n$ (3.1). Then if the equilibrium x_0 is stable with respect to the functions $f_i(x) \in C(\mathbf{R}^n)(i = 1, \dots, k, k, < n)$, then for any sufficiently small $\varepsilon > 0$ a trajectory of system (3.1) is obtained, a certain section of which belongs to the set

$$\{x: \; f_i(x) = f_i(x_0), \; i = 1, \dots, k; \; \rho(x_0, x) < \varepsilon\}$$

Note that in the hyperbolic case this fact follows trivially from the theorem of the topological equivalence to a standard multidimensional saddle (the existence of outgoing and incoming "whiskers"). But this also occurs in the general case when the equilibrium position is unstable.

Proof. Without loss of generality we can assume that $x_0 = 0$ and $f_i(x_0) = f_i(0) = 0$.

We have from the definition of instability that a certain number $\alpha > 0$ exists such that for any number $\delta > 0$ a point x_δ and an instant of time $t(\delta) > 0$ are obtained such that $\|x_\delta\|$, but

$$\| g^{t(\delta)}(x_\delta) \| = \alpha \tag{3.2}$$

where $g^t(x)$ is a trajectory of system (3.1) with initial condition $g^{t=0}(x) = x$.

We will now consider a certain strictly decreasing sequence $\{\delta_n\} \to 0$. Then, for any number $\delta > 0$ a number $N = N(\delta)$ is obtained, which is determined starting from the condition $\delta > \delta_N(\delta_N \to \{\delta_n\})$, such that each of the trajectories $g^t(x_{\delta_n})$ when $n = N(\delta)$ emerges, at the instant $t = 0$. from the point (x_{δ_n}), belonging to the δ-neighbourhood of zero. Suppose we now specify a certain sufficiently small positive number $\varepsilon < \alpha$. Each trajectory $g^t(x_{\delta_n})$ for $n > N(\delta = \varepsilon/4)$ emerges, at the instant $t = 0$, from the $(\varepsilon/4)$-neighbourhood of zero and, at the instant $t(\delta_n)$, intersects the sphere $\rho(x, x_0) = \|x\| = \alpha$ and therefore an instant $t(\delta_n) < (0 < t(\delta_n) < t(\delta_n))$ exists at which this trajectory also intersects the sphere $\|x\| = \varepsilon/2$; we will denote the points of intersection by x_n (if the point of intersection of the trajectory $g^{\tau(\delta_n)}(x_{\delta_n})$ with the sphere $\|x\| = \varepsilon/2$ is not unique, we can take as x_n any of these points, for example, the one at which the trajectory $g^{\tau(\delta_n)}(x_{\delta_n})$, emerging at the instant $t = 0$ from point x_{δ_n}, intersects the sphere $\|x\| = \varepsilon/2$ the first time.

$$g^{\tau(\delta_n)}(x_{\delta_n}) = x_n; \; \| x_n \| = \varepsilon/2$$

We will now consider the sequence $\{x_n\}$ $(n > N(\delta = \varepsilon/4))$. Note that here each of the following two possible versions must be considered separately:

(1) an infinite number of different points n in the sequence $\{x_n\}$.

(2) a finite number of different points in the sequence $\{\bar{x}_n\}$ such that at least one of them say, x^*, has infinite by many "numbers": $x^* = x_{n_1} = x_{n_2} = \dots$.

However, it is easy to see that the proof presented below for the first of these two versions, is also suitable for the case when the second version occurs, so that it makes no sense to consider the latter separately here.

Since the sphere is a compactum in $\mathbf{R}^n$, we can choose from the infinite sequence $\{x_n\}$ of points on the sphere $\|x\| = \varepsilon/2$ a converging subsequence $\{\tilde{x}_m\}$ (a function $n(m)$ exists: $N \to N$: $\tilde{x}_m = x_{n(m)}$, $\forall m$), which converges to a point x^* of this sphere: $\{\tilde{x}_m\}_{m\to\infty} \to x^*_\varepsilon$, $\|x^*_\varepsilon\| = \varepsilon/2$.

We will consider the trajectory of system (3.1) passing through the point x^*, and show that the whole of this trajectory belongs to the surface

$$f_i(x) = f_i(x_0) = 0, \; i = 1, \dots, k$$

We will first assume that a number $i(1 \leqslant i \leqslant k)$ is obtained such that $f_i(x_\varepsilon^*) \neq 0$; suppose, without loss of generality, that $f_i(x_\varepsilon^*) = 2\beta > 0$.

We then have, from the continuity of the functions $f_i(x)$, that a number $1(\beta)$ is obtained such that $\|x - x_\varepsilon^*\| < l \Rightarrow |f_i(x) - f_i(x_\varepsilon^*)| = |f_i(x) - 2\beta| < \beta$, i.e. $f_i(x) > \beta$.

It follows from the convergence $\{\tilde{x}_m\} \to x_\varepsilon^*$ that a number M_1 is obtained such that $\|\tilde{x}_m - x_\varepsilon^*\| < l(\beta)$, $\forall m > M_1$. Therefore

$$f_i(\tilde{x}_m) > \beta, \quad \forall m > M_1 \tag{3.3}$$

We will now specify a certain arbitrary number $\delta > 0$, as small as desired. We will define the number M_2: $n(M_2) > N(\delta)$, where the functions $n(m)$ and $N(\delta)$ are defined above.

Suppose $M = \max(M_1, M_2)$. We then obtain, taking the above and inequality (3.3) into account, that any of the trajectories $g^t(x_{\delta_{n(m)}})$, $m > M$, emerges at the instant $t = 0$ form the point $(x_{\delta_{n(m)}})$, lying in the δ-neighbourhood of zero, and at certain instant $(\tau(\delta_{n(m)}))$ is incident on the point (namely, on $x_{n(m)} = \tilde{x}_m$), at which the value of the function $f_i(x)$ is greater than a certain fixed number $\beta > 0$. This indicates that the zero equilibrium position of system (3.1) is unstable with respect to the function $f_i(x)$, which contradicts the condition. Consequently, our assumption is untrue and $f_i(x_\varepsilon^*) = 0$ for each $i = 1, \dots, k$.

We will now assume that, for a certain number $T > 0$, a number i is obtained such that $f_i(g^T(x_\varepsilon^*)) \neq 0$; suppose (without loss of generality) $f_i(g^T(x_\varepsilon^*)) = 2\beta$, $\beta > 0$. By virtue of the continuity of the function f_i a number $l_1 = l_1(\beta)$ exists such that

$$\| x - g^T(x_\varepsilon^*) \| < l_1 \Rightarrow | f_i(x) - f_i(g^T(x_\varepsilon^*)) | < \beta \Rightarrow f_i(x) > \beta \tag{3.4}$$

By virtue of the theorem of the continuous dependence of the solutions of differential equations on the initial conditions, we conclude that a number $l_2 = l_2(l_1(\beta)) = l_2(\beta)$ is obtained such that

$$\| x - x_\varepsilon^* \| < l_2 \Rightarrow \| g^T(x), g^T(x_\varepsilon^*) \| < l_1 \tag{3.5}$$

We obtain from the convergence $\{\tilde{x}_m\} \to x_\varepsilon^*$ in turn that a number M_3 exists such that

$$\| x_\varepsilon^* - \tilde{x}_m \| < l_2(\beta), \quad \forall m > M_3 \tag{3.6}$$

Hence, by virtue of (3.4)–(3.6) we have: $f_i(g^T(\tilde{x}_m)) > \beta \; \forall m > M_3$. Suppose now $M_4 = \max(M_{3,2})$, where $M_2 = M_2(\delta)$ is defined above. Then, for each $\delta > 0$ as small as desired, any of the trajectories $g^t(x\delta_{n(m)})$, $m > M_4$, emerges at the instant $t = 0$ from the point $(x_{\delta_{n(m)}})$, lying in the δ-neighbourhood of the equilibrium position $x_0 = 0$, and is a certain instant $(t = \tau(\delta_{n(m)} + T))$ is incident on the point (namely, on the point $g^T(x_{n(m)}) = g^T(\tilde{x}_m)$), at which the value of the function $f_i(x)$ is greater than a certain fixed number $\beta > 0$. The latter indicates that the equilibrium $x_0 = 0$ is unstable with respect to the function f_i, which contradicts the assumption.

Hence, the assumption is untrue and

$$f_i(g^T(x_\varepsilon^*)) = 0, \quad \forall T > 0 \quad i = 1, \dots, k$$

Similarly $f_i(g^T(x_\varepsilon^*)) = 0$ for negative T also. As a result we obtain that any trajectory $g^t(x_\varepsilon^*)$ belongs to the set of level

$$\{x: \; f_i(x) = f_i(x_0) = 0, \; i = 1, \dots, k\}$$

Since the point x_ε^* lies in the ε-neighbourhood of the point $x_0 = 0$ ($\|(x_\varepsilon^*\| = \varepsilon/2$, see above), a certain section of the trajectory $g^t(x_\varepsilon^*)$ also lies in this neighbourhood. Hence, for any sufficiently small $\varepsilon > 0$ specified in advance a trajectory of system (3.1) exists which belongs to the set

$$\{x: \; f_i(x) = f_i(x_0), \; i = 1, \dots, k; \; \rho(x, x_0) < \varepsilon\}$$

which proves Proposition 1.

Note that in the case when the sequence $\{x_n\}$ consists of a finite number of points, so that one of them, say, x^*, has infinitely many "number" $x_\varepsilon^* = x_{n1} = x_{n2} = \dots$ $(\{n_k\} \to \infty)$, the trajectory $g^t(x_\varepsilon^*)$ approaches infinitesimally close to the point $x_0 = 0$ as $t \to -\infty$, and hence the proof is simplified here:

the fact that all the functions $f_i(x)$ vanish at points of this trajectory $g^t(x_\varepsilon)$ immediately follows from the definition of the stability of equilibrium position x_0 with respect to the functions $f_i(x)$ $(i = 1, \ldots, k)$.

Suppose now that the functions $f_i(x)$ are of the class C^m, $m \geq 1$ or are analytic; $d^{(p)}f(v)$ is the pth Lie derivative with respect to time of the function f, by virtue of the equations of motion (3.1).

Corollary 1. If the unstable equilibrium position $x_0 = 0$ of system (3.1), $v_j(x) \in C^{m-1}(\mathbf{R}^n)(j = 1, \ldots, n;$ $m \geq 1)$ is stable with respect to each of the functions $f_1(x), \ldots, f_k(x)$, $f_i \in C^m(\mathbf{R}^n)$, then for each $\varepsilon > 0$ as small as desired in $\mathbf{R}^n$ a curve exists, all the points of which satisfy the algebraic system

$$d^p f_i(x)/dt^p = d^p f_i(v)(x) = 0, \quad p = 0, 1, \ldots, m; \quad i = 1, \ldots, k \tag{3.7}$$

and such that a certain section of this curve lies in the ε-neighbourhood of the point $x_0 = 0$.

Corollary 1 in fact immediately follows from Proposition 1.

The trajectory $g^t(x_\varepsilon^*)$, defined in the proof of Proposition 1, belongs to the set $\{x: f_i(x) = 0\}$. Suppose at a certain point $\tilde{x}_1$ of the trajectory we have $g^t(x_\varepsilon^*) df_i(v)(\tilde{x}_1) = \langle \nabla f_i, v \rangle (\tilde{x}_1) = c > 0$. Then, from the continuity in a certain γ-neighbourhood of the point $\tilde{x}_1$ on the trajectory $g^t(x_\varepsilon^*)$ we have $df_i(v)(x) > c/2$, $\|x - \tilde{x}_1\| < \gamma$. Suppose τ is the time, after which the trajectory, emerging from the point $\tilde{x}_1$, is incident on the point $\tilde{x}_2 = g^t(x_\varepsilon^*) \cap \{x: \|x - \tilde{x}_1\| < \gamma\}$. We have

$$f_i(\tilde{x}_2) = f_i(\tilde{x}_1) + \int_0^\tau df_i(v)(g^t(\tilde{x}_1))dt > f_i(\tilde{x}_1) + \frac{c}{2}\tau = \frac{c}{2}\tau \quad (f_i(\tilde{x}_1) = 0)$$

Hence, $f_i(\tilde{x}_2) \neq 0$. We have obtained a contradiction; consequently, all the points of the trajectory $g^t(x_\varepsilon^*)$ belong to the set $\{x: df_i(v)(x) = 0$ (for each $i = 1, \ldots, k)\}$. Proceeding in exactly the same way for the function $f_i^{(1)}(x) = df_i(v)(x)$ we obtain that all the points of the trajectory $g^t(x_\varepsilon^*)$ belong to the set

$$\{x: df_i^{(1)}(v)(x) = d^{(2)} f_i(v)(x) = 0, \ i = 1, \ldots, k\}$$

etc.

As a result we have that all the points of the trajectory $g^t(x_\varepsilon^*)$ satisfy system (3.7). Further, the point x_ε^* lies inside the ε-neighbourhood of zero ($\|x_\varepsilon^*\| = \varepsilon/2$, see above), so that a certain section of the trajectory $g^t(x_\varepsilon^*)$ also belongs to this neighbourhood. This also proves Corollary 1.

Hence, a certain development of the theorem on y-stability [5] gives the necessary conditions for this stability. However, these conditions are only applicable to autonomous systems and only in the case when the Lyapunov instability of the equilibrium position is known in addition, whereas the theorem of y-stability is applicable in the general case of non-autonomous systems (see Section 3), regardless of whether the nature of the Lyapunov stability of their equilibrium positions is known in advance or not.

Proposition 2. Suppose, in a certain neighbourhood of the point $x_0 = 0$, which is a Lyapunov-unstable equilibrium position of system (3.1), $v_j(x) \in C^{m-1}(\mathbf{R}^n)$, $j = 1, \ldots, n$; $m \geq 1$, the algebraic system

$$d^p f^i / dt^p = d^{(p)} f_i(v)(x) = 0, \quad p = 0, \ldots, m, \quad i = 1, \ldots, k, \quad k < n$$

where $f_i \in C^m(\mathbf{R}^n)$, has a unique solution. Then equilibrium position x_0 is unstable with respect to the set of functions $f_1(x), \ldots, f_k(x)$.

In fact, the assertion formulated in Proposition 2, is a direct consequence of Corollary 1. Note that the conditions of Proposition 1 are constructive. Then, when the condition of Proposition 2 is satisfied, they are not necessarily satisfied (whence it immediately follows that x_0 is unstable with respect to the function f). But these conditions may not be satisfied, obviously, when the condition of Proposition 2 is not satisfied. However, the latter turns out, formally speaking, to be satisfied "almost always", if $m \geq n$, in connection with which we can here make the following remark: For any dynamical system (3.1), $v_j(x) \in C^{n-1}(\mathbf{R}^n)$, $j = 1, \ldots, n$, of "general position" with a Lyapunov-unstable equilibrium position $x_0 = 0$, in the general case of an arbitrary function of the phase variables $f(x) \subset C^n(\mathbf{R}^n)$ $(f(0) = 0)$, instability of the equilibrium position x_0 with respect to the function f will also occur.

In fact, it is easy to show that in the general case of an arbitrary function f of the phase variables of system (3.1), the Jacobian

$$\det\left|\frac{D(f^{(0)}, f^{(1)}(x), \ldots, f^{(n-1)}(x))}{D(x)}\right|_{x_0=0} \neq 0$$

where $f^{(k)}(x) \stackrel{\text{def}}{=} d^k f(x)/dt^k$ from the left-hand side of the kth equation of system (3.7).

Hence, taking Proposition 2 into account, we immediately obtain the assertion formulated in the remark.

Proposition 2 is the sufficient condition for partial instability for autonomous systems with Lyapunov-unstable equilibrium positions. It is easy too show that, in the autonomous case, when the conditions of Rumyantsev's theorem on y-instability are satisfied ($y = (y_1, \dots, y_s)$, $s < n$) [5, 6], the conditions of Proposition 2 will also be satisfied if the function $V(t, y)$, which satisfies the y-instability theorem, is explicitly independent of t.

In fact, in this case the conditions of Proposition 2 are also satisfied for $k = 1$, $m = 1$ and for the function $f_1(x) = f(x) = \tilde{V}(y)$, where $\tilde{V}(y)$ is any function, identical with the function $V(y)$ in the region $\tilde{D} \subset \mathbf{R}^s_{\{y\}}$, if D is the region $V(y) > 0$, which occurs in the y-instability theorem, and of a certain positive function of the class $C^1(\mathbf{R}^s_{\{y\}})$, continued into the region $B_{\varepsilon 0}\backslash\bar{D}$ ($\varepsilon_0 > 0$), preserving the derivative on the boundary ∂D. And from the instability of the equilibrium position x_0 with respect to a certain function $V(y)$ it also follows that it is unstable with respect to the set of variables $\{y\}$.

However, if the function $V(t, y)$, which satisfies the y-instability theorem, depends explicitly on t, this does not imply any conclusions regarding the satisfaction of the conditions of Proposition 2. Note also that in order to be able to use Proposition 2 in each specific case it is necessary, in addition, to have available information on the Lyapunov instability of the equilibrium position, whereas the y-instability theorem automatically gives the sufficient conditions for Lyapunov instability, in addition to the sufficient conditions for y-instability (as it should do).

In the problem of stability with respect to a part of the variables (a, b) of the equilibrium positions of system (1.1)–(1.4), corresponding to the Riemann ellipsoids, the situation of a general position also arises, with which we shall deal in the next observation. We will show this.

4. THE STABILITY OF RIEMANN ELLIPSOIDS IN THE SENSE OF LYAPUNOV'S DEFINITION (1.11)

We will consider the problem of the partial stability with respect to the variables a and b of the equilibrium positions of system (1.1)–(1.4), corresponding to Riemann ellipsoids.

It was shown earlier [7], that system (1.1)–(1.4) can be reduced [7] by the diffeomorphism

$$(a, b, \dot{a}, \dot{b}, p, q, r, \omega_1, \omega_2, \omega_3) \to \{z\} = \{a, b, p_{(a)}, p_{(b)}; G_1, G_2, G_3, l_1, l_2, l_3\} \tag{4.1}$$

specified by the relations

$$G_1 = (M/5)((b^2 - c^2)^2)/(b^2 + c^2))p + (4M/5)((b^2c^2)/(b^2 + c^2))\omega_1 \quad (123,\ abc,\ pqr)$$
$$l_1 = -(2M/5)bc\omega_1 \quad (123,\ abc)$$
$$p_a = (M/5)\ (\dot{a}(1 + 1/(a^4b^2)) + \dot{b}/(a^3b^3))$$
$$p_{(b)} = (M/5)(\dot{b}(1 + 1/(a^2b^4)) + \dot{a}/(a^3b^3))$$

to a system of Hamiltonian form

$$\dot{z} = \{z, H\} \tag{4.2}$$

Here $H(z)$ is the energy (1.7), expressed in $\{z\}$ coordinates (4.1)

$$H = \mathcal{K}|_{c=1/(ab)} + 5(2M)^{-1}(a^4b^4(p_{(a)}^2 + p_{(b)}^2) + (ap_{(a)} - bp_{(b)})^2)(a^4b^4 + a^2 + b^2)^{-1} \tag{4.3}$$

where

$$\begin{aligned}\mathcal{K}(z) = {} & 5(2M)^{-1}[(b^2 + c^2)(b^2 - c^2)^{-2}G_1^2 + (a^2 + c^2)(a^2 - c^2)^{-2}G_2^2 + \\ & + (a^2 + b^2)(a^2 - b^2)^{-2}G_3^2 + 4bc(b^2 - c^2)^{-2}G_1l_1 + 4ac(a^2 - c^2)^{-2}G_2l_2 + \\ & + 4ab(a^2 - b^2)^{-2}G_3l_3 + (b^2 + c^2)(b^2 - c^2)^{-2}l_1^2 + (a^2 + c^2)(a^2 - c^2)^{-2}l_2^2 + \\ & + (a^2 + b^2)(a^2 - b^2)^{-2}l_3^2] + W\end{aligned} \tag{4.4}$$

In Eq. (4.2) {,} is the Poisson bracket in the space of functions $C^\infty(Z)$, specified as follows:

$$\{G_i, G_j\} = \varepsilon_{kji} G_k, \quad \{l_i, l_j\} = \varepsilon_{kji} l_k, \quad \{a, p_{(0)}\} = \{b, p_{(b)}\} = 1 \tag{4.5}$$

and the Poisson brackets between all the remaining pairs of phase variables are identically equal zero.

Everywhere henceforth we will consider system (1.1)–(1.4) in $\{z\}$ coordinates (4.1), i.e. in the form (4.2), since this is more convenient from the computational point of view.

Suppose now that z_0 is an equilibrium position of system (1.1)–(1.4). We consider system (1.1)–(1.4) in the set of the level of integrals of the momentum (1.8) and the circulation (1.9)

$$M_{z_0} = \{z : \sum G_i^2 = G_i^2(z_0),\ \sum l_i^2 = l_i^2(z_0)\}$$

$$M_{z_0} = M_{z_0}\{\tilde{z}\}, \quad (\tilde{z}) = (a, b, p_{(a)}, p_{(b)}; G_1, l_1, G_2, l_2)$$

We have the system

$$\dot{\tilde{z}} = \{\tilde{z}, h(\tilde{z}; (z_0))\}' \tag{4.6}$$

(which is strictly Hamiltonian [7]).

Here $h(\tilde{z}; (z_0))$ is the limitation of the function (4.3) on the level M_{z_0}, and the notation (z_0) indicates parametric dependence of the function $h(\tilde{z}; (z_0))$ on the coordinates of the point z_0 in the space of the set of equilibrium positions considered, corresponding to the equilibrium ellipsoids; the bracket $\{,\}'$ is the limitation of the Poisson bracket {,} in the space $C'^\infty(M_{z_0})$.

As will be shown below, for almost each Riemann ellipsoid z_0 of the first and second families in a certain neighbourhood $O_{(z_0)}$ of the point z_0 in phase space $(G_i, l_i, p_{(a)}, p_{(b)}, a, b, i = 1, 2)$ of system (4.6), corresponding to this ellipsoid z_0, the solution of the corresponding algebraic system (3.7) with $f_1 = a$ and $f_2 = b$ is unique: $z = \tilde{z}_0$.

Then, taking into account Proposition 2 from Section 3 (and also the fact that for each unstable equilibrium position $\tilde{z}_0$ of system (4.6) there is obviously an unstable equilibrium position z_0 of system (4.2) and conversely, for each unstable equilibrium position z_0 of system (4.2) the corresponding equilibrium position of system (4.6) z_0 is also unstable) we obtain that almost all the unstable equilibrium positions of system (1.1)–(1.4) will at the same time also be unstable with respect to the variables (a, b). And of course, by virtue of the discussion in Section 2, the corresponding equilibrium ellipsoids will certainly be unstable figures of equilibrium in the sense of Lyapunov's definition (1.11).

The stability with respect to the semi-axes of the ellipsoids of the first family of Riemann ellipsoids. Riemann showed [3], that equilibrium ellipsoids only exist in two cases:

(1) when one of the three pairs (ω_1, p), (ω_2, q) or (ω_3, r) (or (G_i, l_i), $i = 1, 2, 3$) has both zero components (without loss of generality – the first: $l_1 = G_1 = 0$),

(2) when two of these pairs have zero components ($l_1 = G_1 = 0$ and $l_2 = G_2 = 0$).

We will consider the first of these families of equilibrium ellipsoids P_1^2. It is formed by the ellipsoids

$$z_0 = \{a = a_0, b = b_0, G_1 = l_1 = 0, G_2 = G_{20}, l_2 = l_{20}, G_3 = G_{30}, l_3 = l_{30}\} \tag{4.7}$$

in the specification of which the six parameters are related [3] by four equilibrium equations.

In the parametric space of the family P_1^2 of ellipsoids (4.7) we will choose (a_0, b_0) as coordinates [3].

Taking into account the form of the function $h(\tilde{z}; (z_0))$ and the Poisson bracket $\{,\}'$, we see that all points which satisfy system (3.7) for dynamical system (4.60 and for $f_1 = a$ and $f_2 = b$, necessarily satisfy the system

$$\begin{cases} a = a_0, \ b = b_0, \ p_{(a)} = p_{(b)} = 0 \\ h_a(\tilde{z}; (z_0)) = \dot{p}_{(a)} = 0, \ h_b(\tilde{z}; (z_0)) = \dot{p}_{(b)} = 0 \\ \dfrac{d}{dt}(h_a(\tilde{z}; (z_0))) = 0, \ \dfrac{d}{dt}(h_b(\tilde{z}; (z_0)) = 0 \end{cases} \tag{4.8}$$

Here and henceforth we will use the notation $h_a = \partial h/\partial a$, $h_b = \partial h/\partial b$

System (4.8) is equivalent to a system of four algebraic equations

$$h_a(\tilde{z}; (z_0))|_* = h_b(\tilde{z}; (z_0))|_* = 0 \tag{4.9}$$

$$\left[\frac{d}{dt}(h_a(\tilde{z};(z_0)))\right]\Bigg|_* = \left[\frac{d}{dt}(h_b(\tilde{z};(z_0)))\right]\Bigg|_* = 0$$

with four unknowns: G_1, l_1, G_2 and l_2. Here and henceforth the asterisk denotes that the value of the derivative is taken at the point with coordinates $a = a_0$, $b = b_0$ and $p_{(a)} = p_{(b)} = 0$.

We will show that for almost all values of z_0 the solution of system (4.9) $G_i = G_{i0} = G_i(z_0)$, $l_i = l_i(z_0)$ $(i = 1, 2)$ is unique in a certain neighbourhood of the point $(G_1(z_0), G_2(z_0), l_1(z_0), l_2(z_0))$.

We will write Eqs (4.9) in explicit form, taking into account the explicit form of the function $h(\tilde{z}'(z_0))$ (the two asterisks here and henceforth denote that the corresponding derivatives are taken at the point $(a, b) = (a_0 b_0)$)

$$\psi_1 = \frac{5}{2M}\left[\frac{\partial}{\partial a}\left(\frac{b^2+c^2}{(b^2-c^2)^2}\right)\Bigg|_{**} G_1^2 + \frac{\partial}{\partial a}\left(\frac{b^2+c^2}{(b^2-c^2)^2}\right)\Bigg|_{**} l_1^2 + \frac{\partial}{\partial a}\left(\frac{4bc}{(b^2-c^2)^2}\right)\Bigg|_{**} G_1 l_1 + \right.$$

$$+\frac{\partial}{\partial a}\left(\frac{a^2+c^2}{(a^2-c^2)^2}\right)\Bigg|_{**} G_2^2 + \frac{\partial}{\partial a}\left(\frac{4ac}{(a^2-c^2)^2}\right)\Bigg|_{**} G_2 l_2 + \frac{\partial}{\partial a}\left(\frac{a^2+c^2}{(a^2-c^2)^2}\right)\Bigg|_{**} l_2^2 +$$

$$+\frac{\partial}{\partial a}\left(\frac{a^2+b^2}{(a^2-b^2)^2}\right)\Bigg|_{**} (G_0^2 - G_1^2 - G_2^2) + \frac{\partial}{\partial a}\left(\frac{4ab}{(a^2-b^2)^2}\right)\Bigg|_{**} (G_0^2 - G_1^2 -$$

$$\left. -G_2^2)^{1/2}(l_0^2 - l_1^2 - l_2^2)^{1/2} + \frac{\partial}{\partial a}\left(\frac{a^2+b^2}{(a^2-b^2)^2}\right)\Bigg|_{**} (l_0^2 - l_1^2 - l_2^2)\right] + \frac{\partial}{\partial a}(W(a,b))|_{**} = 0 \tag{4.10}$$

$$\psi_2 = \psi_1 |_{\partial/\partial a \to \partial/\partial b} = 0 \tag{4.11}$$

$$\psi_3 = B_a^{(1)} + B_a^{(2)} + B_a^{(3)} = 0 \tag{4.12}$$

$$\psi_4 = B_b^{(1)} + B_b^{(2)} + B_b^{(3)} = 0 \tag{4.13}$$

where

$$B_a^{(1)} = \frac{5}{M}\left\{\left(\frac{a_0^2+b_0^2}{(a_0^2-b_0^2)^2} G_3 G_2 + \frac{2a_0 b_0}{(a_0^2-b_0^2)^2} l_3 G_2 - \frac{a_0^2+c_0^2}{(a_0^2-c_0^2)^2} G_2 G_3 - \frac{2a_0 c_0}{(a_0^2-c_0^2)^2} l_2 G_3\right)\otimes\right.$$

$$\otimes\left(\frac{\partial}{\partial a}\left(\frac{b_0^2+c_0^2}{(b_0^2-c_0^2)^2}\right)\Bigg|_{**} 2G_1 + \frac{\partial}{\partial a}\left(\frac{4b_0 c_0}{(b_0^2-c_0^2)^2}\right)\Bigg|_{**} l_1\right) + \left(\frac{a_0^2+b_0^2}{(a_0^2-b_0^2)^2} l_2 l_3 + \frac{2a_0 b_0}{(a_0^2-b_0^2)^2} G_3 l_2 - \right.$$

$$\left. -\frac{a_0^2+c_0^2}{(a_0^2-c_0^2)^2} l_2 l_3 - \frac{2a_0 c_0}{(a_0^2-c_0^2)^2} G_2 l_3\right)\otimes\left(\frac{\partial}{\partial a}\left(\frac{4b_0 c_0}{(b_0^2-c_0^2)^2}\right)\Bigg|_{**} G_1 + \right.$$

$$\left.\left. + \frac{\partial}{\partial a}\left(\frac{b_0^2+c_0^2}{(b_0^2-c_0^2)^2}\right)\Bigg|_{**} 2l_1\right)\right\}\Bigg|_{\substack{g_3 = (G_0^2 - G_1^2 - G_2^2)^{1/2} \\ l_3 = (l_0^2 - l_0^2 - l_1^2 - l_2^2)^{1/2}}}$$

$(123, a_0 b_0 c_0)$

$$B_b^{(i)} = B_a^{(i)} |_{\partial/\partial a \to \partial/\partial b} \tag{4.14}$$

In formula (4.11) the symbol $|_{\partial/\partial a \to \partial/\partial b}$ denotes that an explicit expression for the function ψ_2 is obtained if, in the expression for the function ψ_1 (4.10) all partial derivatives with respect to the variable a are replaced by derivatives with respect to the variable b.

We will now consider the Jacobian of the matrix

$$A=\frac{D(\psi_1,\psi_2,\psi_3,\psi_4)}{D(G_1,l_1,G_2,l_2)} \tag{4.15}$$

where all the derivatives are taken at the point $G_i = G_{i0}$, $l_i = l_{i0}$ $(i = 1, 2)$.

Differentiating the functions ψ_1 and ψ_2 with respect to the variables G_1 and l_1, we obtain that the derivatives each represent the sum of terms proportional either to G_1 or l_1, so that

$$\partial\psi_i/\partial G_1|_0=\partial\psi_i/\partial l_1|_0=0,\quad i=1,2$$

The zero subscript denotes that the derivatives are taken at the point $(G_1, l_1, G_2, l_2) = (0, 0, G_{20}, l_{20})$.

Further, the functions ψ_3 and ψ_4, written on the left-hand sides of Eqs (4.12) and (4.13), are sums of terms of the form $\varphi(a_0, b_0)\, s_1s_2s_3$, where φ is a certain function and s is G or l. Hence, when differentiating the functions ψ_3 and ψ_4 with respect to the variables G_2 and l_2 we obtain functions which are the sums of terms proportional to s_1, i.e. l_1 or G_1. Hence we have

$$\partial\psi_i/\partial G_2|_0=\partial\psi_i/\partial l_2|_0=0,\quad i=3,4$$

The matrix A when $l_{i0} = G_{i0} = 0$ therefore has a partitioned form and its determinant (4,15) will be equal to the product of determinants

$$\det A=\det A_1\times\det A_2;\qquad A_1=\left.\frac{D(\psi_1,\psi_2)}{D(G_2,l_2)}\right|_0,\qquad A_2=\left.\frac{D(\psi_3,\psi_4)}{D(G_1,l_1)}\right|_0 \tag{4.16}$$

We will first consider the matrix A_1.

Taking the relation $\partial G_3/\partial G_2 = -G_2/G_3$ into account, we obtain

$$\left.\frac{\partial\psi_1}{\partial G_2}\right|_0=2G_2\left\{\frac{\partial}{\partial a}\left(\frac{a^2+c^2}{(a^2-c^2)^2}\right)\right|_{**}-\frac{\partial}{\partial a}\left(\frac{a^2+b^2}{(a^2-b^2)^2}\right)\bigg|_{**}+\frac{2l_2\partial}{G_2\partial a}\left(\frac{ac}{(a^2-c^2)^2}\right)\bigg|_{**}-$$
$$\left.\frac{2l_3\partial}{G_3\partial a}\left(\frac{ab}{(a^2-c^2)^2}\right)\right|_{**}\Bigg\}$$

Similarly

$$\partial\psi_2/\partial G_2|_0=\partial\psi_1/\partial G_2|_{\partial/\partial a\to\partial/\partial b}$$

In exactly the same way

$$\partial\psi_1/\partial l_2|_0=\partial\psi_1/\partial G_2|_{l_i\leftrightarrow G_i},\quad i=2,3\ \text{ and }\ \partial\psi_2/\partial l_2|_0=\partial\psi_1/\partial l_2|_{\partial/\partial a\to\partial/\partial b}$$

Hence the elements of the matrix considered are

$$\left.\frac{\partial\psi_1}{\partial G_2}\right|_0=2G_{20}\frac{\partial J_1}{\partial a},\ \left.\frac{\partial\psi_2}{\partial G_2}\right|_0=2G_{20}\frac{\partial J_1}{\partial b} \tag{4.17}$$
$$\left.\frac{\partial\psi_1}{\partial l_2}\right|_0=2l_{20}\frac{\partial J_2}{\partial a},\ \left.\frac{\partial\psi_2}{\partial l_2}\right|_0=2l_{20}\frac{\partial J_2}{\partial b}$$

where

$$J_1=J_1\left(a,b,\frac{l_{20}}{G_{20}},\frac{l_{30}}{G_{30}}\right)=\left\{\frac{a^2+c^2}{(a^2-c^2)^2}-\frac{a^2+b^2}{(a^2-b^2)^2}+\frac{2l_{20}}{G_{20}}\frac{ac}{(a^2-c^2)^2}-\frac{2l_{30}}{G_{30}}\frac{ab}{(a^2-b^2)^2}\right\}\Bigg|_{**},$$
$$J_2=J_1|_{G_i\leftrightarrow l_i(i=1,2)}$$

Hence, we obtain

$$\det A_1 = 4G_{20}l_{20}\det D(J_1, J_2)/D(a,b)|_{**} \tag{4.18}$$

We note now that the functions J_1 and J_2 are identical [3] with the left-hand sides of the first two equilibrium equations. From these equations, for any l_{20}/G_{20} and l_{30}/G_{30}, the parameters a_0 and b_0 of the Riemann ellipsoid are uniquely defined [3] for the given values of the parameters l_{20}/G_{20} and l_{30}/G_{30}, which are implicit functions of $a_0(l_{20}/G_{20}, l_{30}/G_{30})$, $b_0(l_{20}/G_{20}, l_{30}/G_{30})$. Hence

$$\det D(J_1, J_2)/D(a,b)|_{**} \neq 0 \tag{4.19}$$

which can, of course, also be obtained by direct calculation.

We will now consider the matrix A_2 from formula (4.16).

Starting from the third equation of (4.9) and the form of the function $h(\tilde{z}; (z_0))$, we have

$$\psi_3(G_1, G_2, l_1, l_2) = \left(\frac{d}{dt}\left(\frac{\partial h}{\partial a}\right)\right)\bigg|_* = \frac{d}{dt}\left(\frac{\partial \mathcal{H}}{\partial a}\right)\bigg|_* = \left(\frac{d}{dt}\frac{\partial \mathcal{H}}{\partial a}\bigg|_{**}\right) + \dot{a}\frac{\partial^2 \mathcal{H}}{\partial a^2}\bigg|_* + \dot{b}\frac{\partial^2 \mathcal{H}}{\partial a \partial b}\bigg|_* = \frac{d}{dt}\left(\frac{\partial \mathcal{H}}{\partial a}\bigg|_{**}\right) =$$

$$= \frac{d}{dt}\psi_1(G_1, G_2, l_1, l_2) = \left(\frac{\partial \psi_1}{\partial G_1}\dot{G}_1\right)\bigg|_{**} + \left(\frac{\partial \psi_1}{\partial G_2}\dot{G}_2\right)\bigg|_{**} + \left(\frac{\partial \psi_1}{\partial l_1}\dot{l}_1\right)\bigg|_{**} + \left(\frac{\partial \psi_1}{\partial l_2}\dot{l}_2\right)\bigg|_{**} \tag{4.20}$$

where $\mathcal{H}$ is the function (4.4), while

$$\dot{G}_1 = \frac{5}{M}\left[\left(\frac{a^2+b^2}{(a^2-b^2)^2}(G_0^2 - G_1^2 - G_2^2)^{1/2} + \frac{2ab}{(a^2-b^2)^2}(l_0^2 - l_1^2 - l_2^2)^{1/2}\right)G_2 - \right.$$

$$\left. -\left(\frac{a^2+c^2}{(a^2-c^2)^2}G_2 + \frac{2ac}{(a^2-c^2)^2}l_2\right)(G_0^2 - G_1^2 - G_2^2)^{1/2}\right] \quad (123, \quad abc) \tag{4.21}$$

Differentiating the function ψ_3 (4.20), taking formulae (4.21) into account, we have

$$\frac{\partial \psi_3}{\partial G_1}\bigg|_0 = \left(\frac{\partial \psi_1}{\partial G_2}\right)\left(\frac{\partial \dot{G}_2}{\partial G_1}\right)\bigg|_0 + \left(\frac{\partial \psi_1}{\partial l_2}\right)\left(\frac{\partial \dot{l}_2}{\partial G_1}\right)\bigg|_0 \tag{4.22}$$

Relation (4.22) follows from the fact that

$$\dot{s}_i|_0 = \dot{s}_i|_{\tilde{z}_0} = 0, \quad i = 1,2, \quad s = G, l$$

since $\tilde{z}_0 = \{a = a_0, b = b_0, G_1 = l_1 = 0, G_{20}, l_{20}\}$ is the equilibrium position of system (4.6) and

$$\frac{\partial \psi_1}{\partial G_1}\bigg|_0 = \frac{\partial \psi_1}{\partial l_1}\bigg|_0 = \frac{\partial \psi_2}{\partial G_1}\bigg|_0 = \frac{\partial \psi_2}{\partial l_1}\bigg|_0 = 0$$

as a consequence of the form of the function ψ_1 (4.10) and ψ_2 (4.11).

We similarly have

$$\frac{\partial \psi_4}{\partial G_1}\bigg|_0 = \left[\left(\frac{\partial \psi_2}{\partial G_2}\right)\left(\frac{\partial \dot{G}_2}{\partial G_1}\right) + \left(\frac{\partial \psi_2}{\partial l_2}\right)\left(\frac{\partial \dot{l}_2}{\partial G_1}\right)\right]\bigg|_0 \tag{4.23}$$

Further, in exactly the same way

$$\frac{\partial \psi_\varkappa}{\partial l_1}\bigg|_0 = \left[\left(\frac{\partial \psi_{\varkappa-2}}{\partial G_2}\right)\left(\frac{\partial \dot{G}_2}{\partial l_1}\right) + \left(\frac{\partial \psi_{\varkappa-2}}{\partial l_2}\right)\left(\frac{\partial \dot{l}_2}{\partial l_1}\right)\right]\bigg|_0, \quad \varkappa = 3,4 \tag{4.24}$$

From relations (4.22)–(4.24) we immediately obtain that

$$[D(\psi_3,\psi_4)/D(G_1,l_1)]|_0=[D(\psi_1,\psi_2)/D(G_2,l_2)]|_0\, B \tag{4.25}$$

where $B = ||\beta_{ij}||$ is a 2×2 matrix, whose elements have the form

$$\beta_{11}=\frac{5}{M}\left(\frac{b_0^2+c_0^2}{(b_0^2-c_0^2)^2}G_{30}-\frac{a_0^2+b_0^2}{(a_0^2-b_0^2)^2}G_{30}-\frac{2a_0b_0}{(a_0^2-b_0^2)^2}l_{30}\right)$$

$$\beta_{12}=\frac{5}{M}\frac{2b_0c_0}{(b_0^2-c_0^2)^2}l_{30},\quad \beta_{22}=\beta_{11}|_{l_{30}\leftrightarrow G_{30}},\quad \beta_{21}=\beta_{12}|_{l_{30}\leftrightarrow G_{30}} \tag{4.26}$$

The condition $\det B = 0$, taking expressions (4.26) into account, is the quadratic equations

$$x^2[(2a_0b_0(a_0^2-b_0^2)^{-2}((b_0^2+c_0^2)(b_0^2-c_0^2)^{-2}-(a_0^2+b_0^2)(a_0^2-b_0^2)^{-2}))]+x[4b_0^2c_0^2(b_0^2-c_0^2)^{-4}-$$

$$-4a_0^2b_0^2(a_0^2-b_0^2)^{-4}-((b_0^2+c_0^2)(b_0^2-c_0^2)^{-2}-(a_0^2+b_0^2)(a_0^2-b_0^2)^{-2})^2)]+$$

$$+2a_0b_0(a_0^2-b_0^2)^{-2}((b_0^2+c_0^2)(b_0^2-c_0^2)^{-2}-(a_0^2+b_0^2)(a_0^2-b_0^2)^{-2})=0 \tag{4.27}$$

in $x = G_{30}/l_{30}$. However, it follows from the equilibrium equations [3] that the quantity G_{30}/l_{30} satisfies another quadratic equation, and it is easy to show that, for almost all a_0 and b_0 neither this equation nor Eq. (4.27) have common roots.

This indicates that, for almost all values of (a_0, b_0) the determinant of the matrix A is non-zero. Hence, taking expression (4.25) and inequality (4.19) into account, we obtain

$$\det A_2 \neq 0 \tag{4.28}$$

As a result we have, from relations (4.16), (4.18), (4.19) and (4.28), that $\det A \neq 0$ for almost all values of a_0 and b_0.

Hence, for almost all ellipsoids of the first family (4.7) the solution $G_1 = l_1 = 0$, $G_2 = G_{20}$, $l_2 = l_{20}$ of algebraic system (4.9) is unique in a certain neighbourhood of the point $(0, 0, G_{20}, l_{20})$.

Consequently, almost all the unstable Riemann ellipsoids of the first family are at the same time unstable with respect to the variables a and b. In other words, almost all the Riemann ellipsoids of the first family with semi-axes (a_0, b_0), belonging to the region U in parameter space $P_2^1\{a_0, b_0\}$, defined in [7], are unstable in the sense of Lyapunov's definition of the stability of the form of equilibrium in the class of perturbations which satisfy Dirichlet's assumptions. And of course, these ellipsoids are unstable figures of equilibrium of the rotating liquid in the sense of the definition given by Lyapunov in [1].

The stability with respect to the semi-axes of the ellipsoids of the second family of Riemann ellipsoids. We will now consider the stability of the equilibrium ellipsoids belonging to the second of the above-mentioned families of Riemann ellipsoids [3, 7]

$$z_0=\{a=a_0, b=b_0, p_{(a)}=p_{(b)}=0,\quad l_1=l_2=G_1=G_2=0, \tag{4.29}$$

$$G_3=\Omega(C_{10}+fG_{20})^{1/2},\quad l_3=-\frac{2M}{5}a_0b_0f\Omega\}$$

$$f\geqslant 1,\quad a_0\geqslant c_0=1/(a_0b_0), b_0>c_0=1/(a_0b_0)$$

The four parameters (a_0, b_0, f, Ω), occurring in the specification of the ellipsoids (4.29), are related [3, 4] by two equilibrium equations.

In the case of ellipsoids (4.29) system (4.6) on the level of the integrals of the momentum (1.8) and circulation (1.9) of system (1.1)–(1.4)

$$\dot{\tilde{z}}=\{\tilde{z},\bar{h}(\tilde{z};(z_0))\} \tag{4.30}$$

has the invariant set [7].

$$N = \{\tilde{z} : G_1 = l_1 = G_2 = l_2 = G_{10} = l_{10} = l_{20} = G_{20} = 0\}$$

The dynamical system defined in this way by system (4.13) on the set N is a Hamiltonian system with phase space $N\ \{(a, b, p_{(a)}, p_{(b)}\}$. The conditions for a certain trajectory of this system to be in the plane $a = a_0$, $b = b_0$ have the form $a = a_0$, $b = b_0$, $p_{(a)} = p_{(b)} = 0$, and hence such a trajectory, that differs from the equilibrium position, does not exist. Consequently, taking Proposition 2 of Section 3 into account, we obtain that any Riemann ellipsoid of the second family, that is unstable (which is essential here) in the class of perturbations which satisfy the following conditions at the initial instant

$$G_i(z(t=0)) = G_{i0}, \quad l_i(z(t=0)) = l_{i0};\ z(t=0) \in N$$

is unstable with respect to the variables a and b.

However, it was shown in [7] that the set of all unstable ellipsoids (4.29) is exhausted by the fact that they are unstable in the class of perturbations belonging to the set N. Hence, we have that all equilibrium positions of system (1.1)–(1.4) of the ellipsoid (4.29) that are Lyapunov-unstable are simultaneously also unstable (in Lyapunov's sense) with respect to the variables a and b.

Consequently, all Riemann ellipsoids of the second family (4.29) with semi-axes which satisfy the opposite condition to that of (3.4.10) from [7] (apart, perhaps, from bifurcation ellipsoids), are unstable forms of equilibrium in the sense of the definition given by Lyapunov [1].

Hence, we obtain that an analysis of the conditional stability has enabled us to answer the question of the nature of the stability of Riemann ellipsoids, understood in accordance with the general Lyapunov formulation.

The stability of equilibrium ellipsoids in the sense of definition (1.11) given by Lyapunov [1], with the exception of special cases of Maclaurin and Jacobi ellipsoids, which were investigated by Lyapunov himself, have not previously been considered in the literature.

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